ON DEGREE SEQUENCE

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Introduction:

A sequence $d_1, d_2, d_3, \ldots, d_n$ of nonnegative integers is called a degree sequence of given graph *G* if the vertices of *G* can be labeled $V_1, V_2, V_3, \ldots, V_n$ so that degree $V_i = d_i$; for all *i* [2]. The sum of the integers $d_1, d_2, d_3, \ldots, d_n$ is equals to $2q$, where *q* is the number of edges of a graph *G*. For a given graph *G*, a degree sequence of *G* can be easily determined [1,3]. Now the question arise, given a sequence $\xi = d_1, d_2, d_3, \ldots, d_n$ of nonnegative integers, then under what conditions does there exist a graph *G*? A necessary and sufficient condition for a sequence to be graphical was found by Havel and later rediscovered by Hakimi [1,2,3]. We have used the theorem of Hakimi S. L. to produce an algorithm, which accepts a sequence and definitely determines whether any simple graph can be drawn for the given degree sequence or not.

Preliminaries:

Definition: A sequence $\xi = d_1, d_2, d_3, \ldots, d_n$ of nonnegative integers is said to be *graphic sequence* if there exists a graph *G* whose vertices have degree d_i and *G* is called *realization* of ξ [1].

For an example:

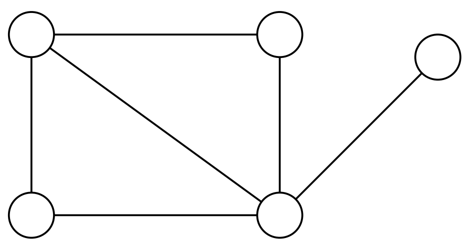

Figure 1:

A degree sequence of the graph (in Figure 1) is 4,3,2,2,1 (or 1,2,2,3,4; or 2,1,4,2,3 etc.). Certainly the conditions:

$d_i \le (n-1)$; for all *i* ………………………………………………………….(1)

and

$\sum d_i =$ even number ………………………………………………………….(2)

are necessary for a sequence to be graphical and should be checked first, but these conditions are not sufficient [2]. A necessary and sufficient condition for a sequence to be graphical was found by Havel and later rediscovered by Hakimi [1,2,3].

For an example, 4,4,3,2,1 is not graphical. Though conditions (1) and (2) satisfies for that sequence, but it is impossible to draw any simple graph for the sequence.

Theorem: A Sequence $\xi = d_1, d_2, d_3, \ldots, d_n$ of nonnegative integers with $d_1 \ge d_2 \ge d_3 \ge \ldots \ge d_n$; $n \ge 2$; $d_1 \ge 1$ is graphical if and only if the sequence $\xi_1 = d_2-1, d_3-1, \ldots, d_{d_1+1}-1, d_{d_1+2}, \ldots, d_n$ is graphical [1,2].

Proof [1,2]**:**

Assume that ξ_1 is a graphical sequence. Then there exists a graph $G_1$of order $n-1$, such that $\xi_1$is the degree sequence of G_1. Thus, the vertices of G_1 can be labeled as $V_2, V_3, \ldots. V_n$; so that

$$\deg(V_i) = \begin{cases} d_i - 1;\ 2 \le i \le d_1+1 \\ d_i;\ d_1+2 \le i \le n \end{cases}$$

A new graph G can be now be constructed by adding a new vertex V_1 and the d_1 edges V_1V_i ; $2 \le i \le d_1+1$. Then in G $\deg(V_i) = d_i$ for $1 \le i \le n$, and so $\xi = d_1, d_2, d_3, \ldots, d_n$ is graphical.

Conversely, let ξ be a graphical sequence. Hence there exist graphs of order n with degree sequence ξ. Among all such graphs let G be one, such that $V(G)=\{V_1, V_2, V_3, \ldots, V_n\}$; $\deg(V_i)=d_i$ for $i=1,2,3,\ldots.,n$ and the $\sum d_i$= even number, the sum of degrees of the vertices adjacent with V_1 is maximum. We show first that V_1 is adjacent with vertices having degrees $d_2, d_3, \ldots, d_{d1+1}$.

Suppose, to the contrary, that V_1 is not adjacent with vertices having degrees $d_2, d_3, \ldots, d_{d1+1}$. Then there exist vertices V_r and V_s with $d_r > d_s$ such that V_1 is adjacent to V_s , but not to V_r . Since, the degree of V_r exceeds that of V_s, there exists a vertex V_t, such that V_t is adjacent to V_r but not to V_s. Removing the degrees V_1V_s and V_rV_t and adding the edges V_1V_r and V_sV_t results in a graph G' having the same degree sequence as G. However, in G' the sum of the degrees of the vertices adjacent to V_1 is larger than that in G, contradicting the choice of G.

Thus, V_1 is adjacent with vertices having degrees $d_2, d_3, \ldots, d_{d1+1}$, and the graph $(G - V_1)$ has degree sequence ξ_1, so ξ_1 is graphical. ♦

Lemma: For a graphic sequence $\xi = d_1, d_2, d_3, \ldots, d_n$ with $d_1 \ge d_2 \ge d_3 \ge \ldots \ge d_n$; $n \ge 2$; $d_1 \ge 1$, if $d_n = 1$, then $(n-1) > d_1$ and $(n-1) > d_2$, where n is number of vertices of the graph G.

Proof:

Let a simple graph $G(V,E)$ with $|V| = n$ number of vertices and $|E| = q$ number of edges. Also considering the degree sequence of the graph G is $\xi = d_1, d_2, d_3, \ldots, d_n$ with $d_1 \ge d_2 \ge d_3 \ge \ldots \ge d_n$; $n \ge 2$; and $d_n = 1$. That means the graph G contains a pendent vertex. Now if we delete the pendent vertex from G, then the modified simple graph G' will contain $(n-1)$ vertices. Then the degree of any vertex of G' will not be $(n-1)$ and can be at most $(n-2)$. Now further adding the pendent vertex back to the graph G' to produce G, causes degree of any one vertex increase by unity. Hence the graph G can have at most one vertex of degree at most $(n-1)$ with one or more pendent vertex. Hence, the lemma proved. ♦

Explanation of the Algorithm with an Example:

Consider a sequence ξ = 4,4,3,3,2 and we have to check whether the sequence ξ is graphic sequence or not.

First of all conditions (1) and (2) are checked and it is satisfied for this ξ. Now,

ξ is sorted in descending order. Deleting 1st term (i.e. 4) and decreasing next 4 terms by 1, we have,

$\xi = \underline{4},4,3,3,2$
to
$\xi_1 = 3,2,2,1$

Similarly proceeding we get,

From $\xi_1 = \underline{3},2,2,1$
to
$\xi_2 = 1,1,0$

From $\xi_2 = \underline{1},1,0$
to
$\xi_3 = 0,0$

We stop here. Since ξ_3 = 0,0.
ξ3 is a sequence representing isolated vertex and it is always possible to draw. Hence ξ_3 is graphic sequence. And hence ξ is graphic sequence. The corresponding graph for ξ is given in the Figure 2.

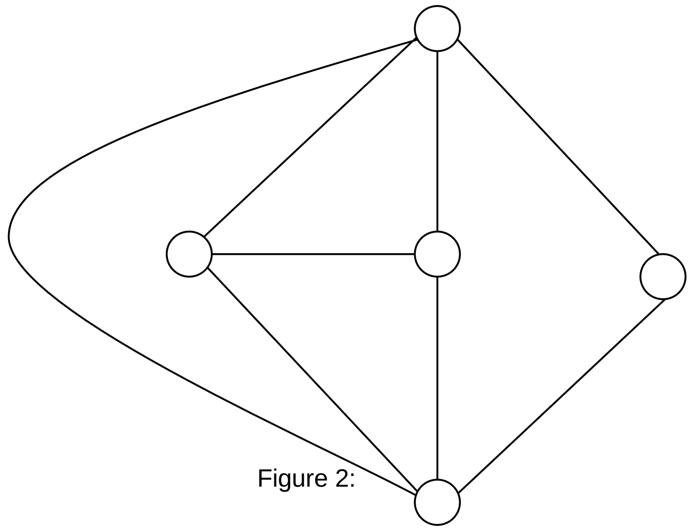

Figure 2:

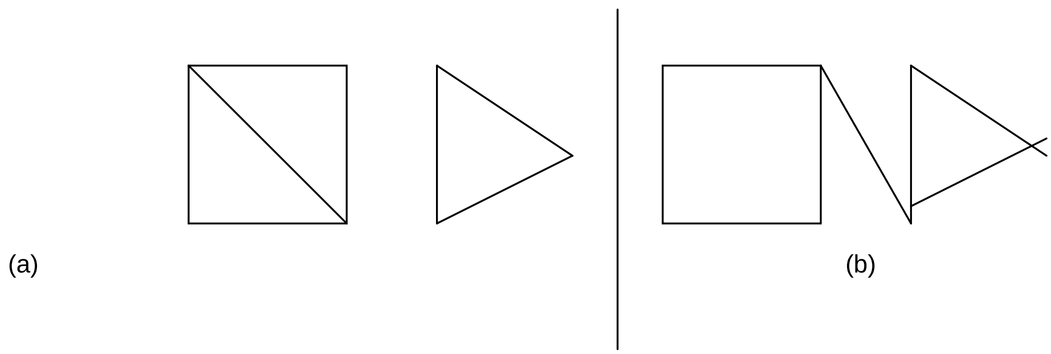

Figure 3: Two Non-Isomorphic graphs, but represents same *degree sequence* ξ = 3,3,2,2,2,2

Algorithm:

Input: Sequence array.
Output: Valid or Invalid sequence.

Main Procedure:
Begin:
Input string length *n*.
Input sequence seq[] array.

Call SortAssend(seq, *n*) procedure.
Call Disp(seq, *n*) procedure.
        flag = Call IsValid(seq, *n*)
If flag = 1 Then
        Output sequence as "Valid Sequence".
Else
        Output sequence as "Invalid Sequence".
End If
Stop.

Procedure: SortAssend(seq, *n*)

```
Begin:
i = 0
L1:     j = 0.
L2:
If seq[j] ≤ seq[j+1] Then
Swap seq[j] and seq[j+1]
End If
j = j+1
If j ≤ n-2-i Then
goto label L2
Else
i = i + 1.
End If
If i ≤ n –1 Then
goto label L1
        End If
Return.
```

Procedure: Disp(seq, n)

```
Begin:
i = 0
L1:
Output seq[i]
i = i + 1
If i ≤ n-1 Then
goto label L1
End If
Return.
```

Procedure: IsValid(seq, n)

```
Begin:
flag = 0, allz = 0, lim = n.
L1:
flag = Call IsNegative(sq, lim)

If flag =1 Then
flag = 0
goto label L3
End If

allz = Call AllZero(sq, lim)

If allz = 1 Then
goto label L3
End If

k = sq[0]
Call ShiftL(sq, lim); i = 0
L2:
Do
sq[i] = sq[i] –1
i = i +1
While i < k goto label L2

lim = lim –1

Call SortAssend(sq, lim)

If flag ≠ 1 allz ≠ 1 Then
goto label L1.
End If
L3:
```

```
If allz =1 Then
        Return allz
    Else
        Return flag
Return.
```

Procedure: ShiftL(seq, lim)

```
Begin:
i = 0
L1:
seq[i]=seq[i+1]
i = i + 1

If i ≤ lim –1 Then
goto label L1
End If
Return.
```

Procedure: IsNegative(seq, lim)

```
Begin:
I = 0
L1:
If seq[i] = 0 Then
    Return 1
i = i +1
End If

If I ≤ lim –1 Then
goto label L1.
End If

Return 0
Return.
```

Procedure: AllZero(seq, lim)

```
Begin:

If seq[i] ≠ 0 Then
    Return 0
i = i + 1
End If

If i ≤ lim –1 Then
    Return 1
End If
Return.
```

Conclusion:

The theorem, proposed by S. L. Hakimi, is recursively used in the algorithm. Any two isomorphic graphs represent the exactly same sequence. However, the converse is not true [1]. Two non-isomorphic graphs represent the same degree sequence (shown in the Figure 3).

Reference:

[1] Arumugam S. and Ramachandran S., Invitation to Graph Theory, Scitech Publications (INDIA) Pvt. Ltd., Chennai, 2002.
[2] Chartrand G. and Lesniak L., Graphs & Digraphs, Third Edition, Chapman & Hall, 1996.
[3] Hartsfield N. and Ringle G., Pearls in Graph Theory A Comprehensive Introduction, Academic Press, INC, Harcourt Brace Jovanovich, publishers, 1997.